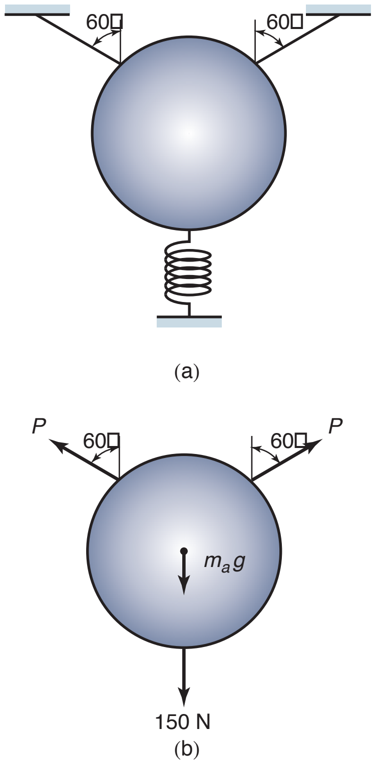


Figure 2.6: Sphere and applied forces. (a) Sphere supported with wires from top and spring at bottom; (b) free-body diagram of forces acting on sphere.

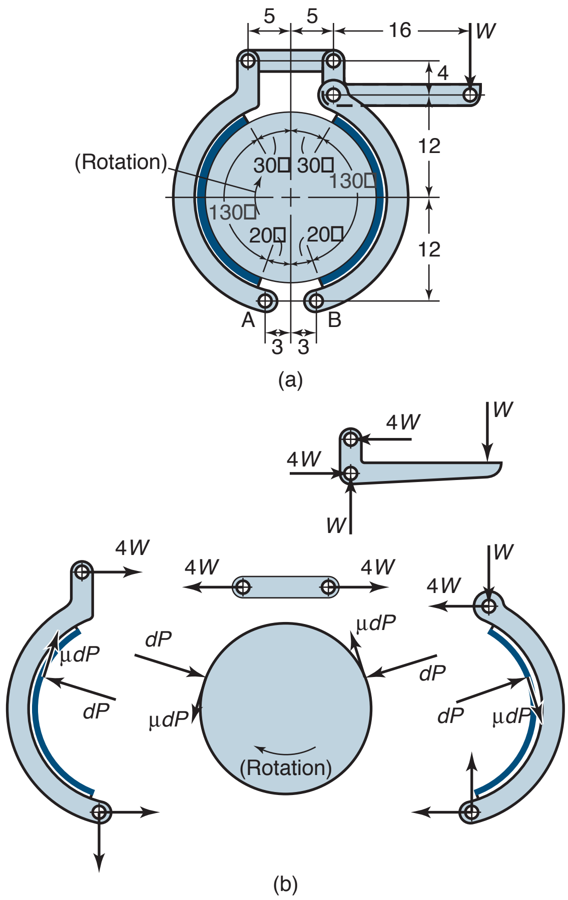


Figure 2.7: External rim brake and applied forces, considered in Example 2.6. (a) External rim brake; (b) external rim brake with forces acting on each part. (Linear dimensions are in millimeters.)

Example 2.6: Free-Body Diagram of an External Rim Brake

Given: The external rim brake shown in Fig. 2.7a.

Find: Draw a free-body diagram of each component of the system.

Solution: Figure 2.7b shows each brake component as well as the forces acting on them. The static equilibrium of each component must be preserved, and the friction force acts opposite to the direction of motion on the drum and in the direction of motion on both shoes. The $4W$ value in Fig. 2.7b was obtained from the moment equilibrium of the lever. Details of brakes are considered in Chapter 18, but in this chapter it is important to be able to draw the free-body diagram of each component.

2.7 Supported Beams

A **beam** is a structural member designed to support loading applied perpendicular to its longitudinal axis. In general, beams are long, often straight bars having a constant cross-section. Often, they are classified by how they are supported. Three major types of support are shown in Fig. 2.8:

1. A **simply supported beam** (Fig. 2.8a) is pinned at one end and roller-supported at the other.

2. A **cantilevered beam** or **cantilever** (Fig. 2.8b) is fixed at one end and free at the other.

3. An **overhanging beam** (Fig. 2.8c) has one or both of its ends freely extending past its supports.

Two major parameters used in evaluating beams are strength and deflection, as discussed in Chapter 5. Shear and bending are the two primary modes of beam loading. However, if the height of the beam is large relative to its width, elastic instability can become important and the beam can twist under loading (see *unstable equilibrium* in Section 9.2.3).

2.8 Shear and Moment Diagrams

Designing a beam on the basis of strength requires first finding its maximum shear and moment. This section describes three common and powerful approaches for developing shear and moment diagrams. Usually, any of these methods will be sufficient to analyze any statically determinate beam, so the casual reader may wish to emphasize one method and then continue to the remaining sections.

2.8.1 Method of Sections

One way to obtain shear and moment diagrams is to apply equilibrium to sections of the beam taken at convenient locations. This allows expression of the transverse shear force, V, and the moment, M, as functions of an arbitrary position, x, along the beam's axis. These shear and moment functions can then be plotted as shear and moment diagrams from which the maximum values of V and M can be obtained.

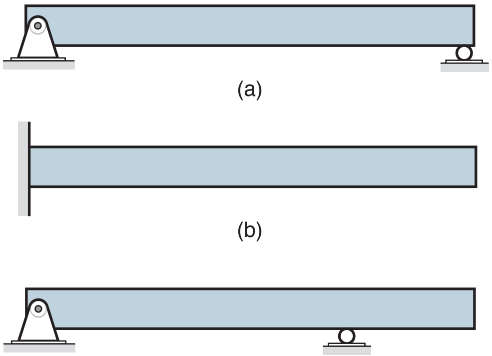


Figure 2.8: Three types of beam support. (a) Simply supported; (b) cantilevered; (c) overhanging.

Design Procedure 2.2: Drawing Shear and Moment Diagrams by the Method of Sections

The procedure for drawing shear and moment diagrams by the method of sections is as follows:

1. Draw a free-body diagram and determine all the support reactions. Resolve the forces into components acting perpendicular and parallel to the beam's axis.

2. Choose a position, x, between the origin and the length of the beam, l, thus dividing the beam into two segments. The origin is chosen at the beam's left end to ensure that any x chosen will be positive.

3. Draw a free-body diagram of the two segments and use the equilibrium equations to determine the transverse shear force, V, and the moment, M.

4. Plot the shear and moment functions versus x. Note the location of the maximum moment. Generally, it is convenient to show the shear and moment diagrams directly below the free-body diagram of the beam.

5. Additional sections can be taken as necessary to fully quantify the shear and moment diagrams.

Example 2.7: Shear and Moment Diagrams by Method of Sections

Given: The bar shown in Fig. 2.9a.

Find: Draw the shear and moment diagrams.

Solution: For $0 \leq x < l/2$, the free-body diagram of the bar section is as shown in Fig. 2.9b. The unknowns V and M are positive. Applying the equilibrium equations gives

$$\sum P_y = 0 \to V = -\frac{P}{2}, \qquad (a)$$

$$\sum M_z = 0 \to M = \frac{P}{2}x. \qquad (b)$$

For $l/2 \leq x < l$, the free-body diagram is shown in Fig. 2.9c. Again, V and M are shown in the positive direction.

$$\sum P_y = 0 \to \frac{P}{2} - P + V = 0, \quad \text{or} \quad V = P/2. \qquad (c)$$

$$\sum M_z = 0 \to M + P\left(x - \frac{l}{2}\right) - \frac{P}{2}x = 0.$$

Therefore,

$$M = \frac{P}{2}(l - x). \qquad (d)$$

The shear and moment diagrams in Fig. 2.9d can be obtained directly from Eqs. (*a*) to (*d*).

2.8.2 Direct Integration

Note that if $q(x)$ is the load intensity function in the y-direction, the transverse shear force is

$$V(x) = -\int_{-\infty}^{x} q(x)\,dx, \qquad (2.4)$$

and the bending moment is

$$M(x) = -\int_{-\infty}^{x} V(x)\,dx = \int_{-\infty}^{x}\int_{-\infty}^{x} q(x)\,dx\,dx. \qquad (2.5)$$

For simple loading cases, direct integration is often the most straightforward method of producing shear and moment diagrams. Since the integral of a curve is its area, graphically producing a shear or moment diagram follows directly from the loading. The only complication arises from point loadings and their use in developing a shear diagram. With concentrated loadings, the shear diagram will take a "jump" equal in magnitude to the applied load. The sign convention used for moment diagrams is important; recall that the sign convention described in Fig. 2.3b is used in this textbook.

Example 2.8: Shear and Moment Diagrams by Direct Integration

Given: The beam shown in Fig. 2.10a. From static equilibrium, it can be shown that R_A = 12 kN and R_B = 4 kN in the directions shown.

Find: The shear and moment diagrams by direct integration. Determine the location and magnitude of the largest shear force and moment.

Solution: The shear diagram will be constructed first. Consider the loads on the beam and work from left to right to construct the shear diagram. The following steps are followed to construct the shear diagram:

1. At the left end (at $x = 0$), there is a downward acting force. As discussed above, this means that the shear diagram will see a jump in its value at $x = 0$. From Eq. (2.4), a downward acting load leads to an upward acting shear force (that is, its sign is opposite to the loading). Thus, the diagram jumps upward by a magnitude of 4 kN.

2. Moving to the right, this value is unchanged until $x = 2$ m, where a 12 kN concentrated load acts upward. This results in a downward jump as shown.

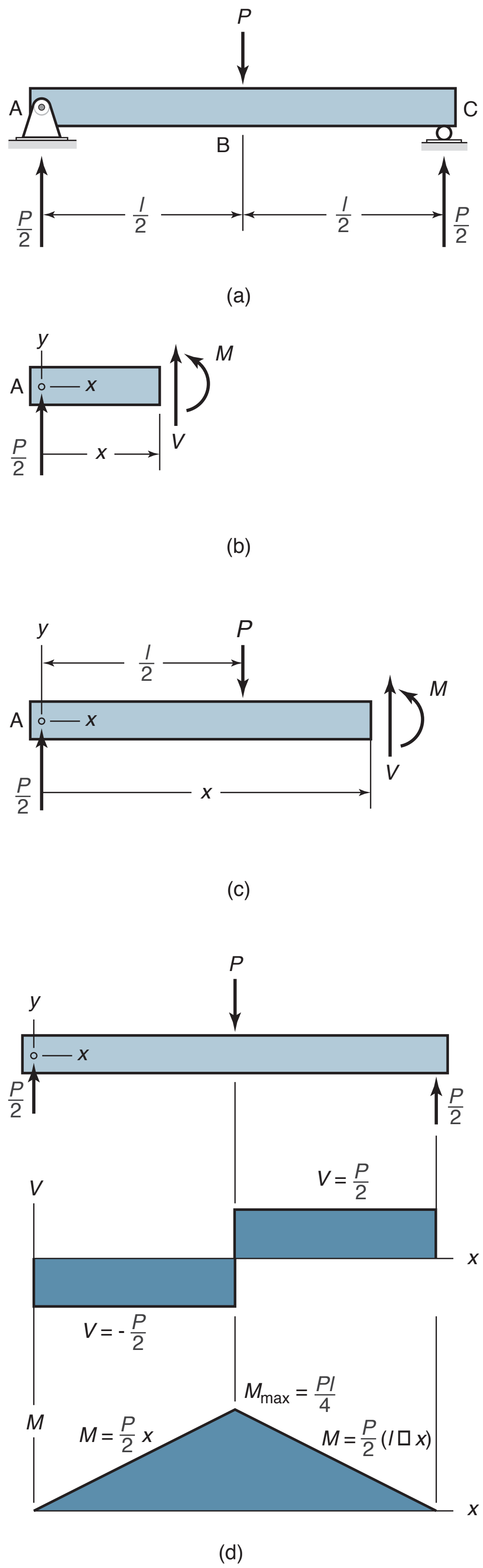


Figure 2.9: Simply supported beam. (a) Midlength load and reactions; (b) free-body diagram for $0 < x < l/2$; (c) free-body diagram for $l/2 \leq x < l$; (d) shear and moment diagrams.

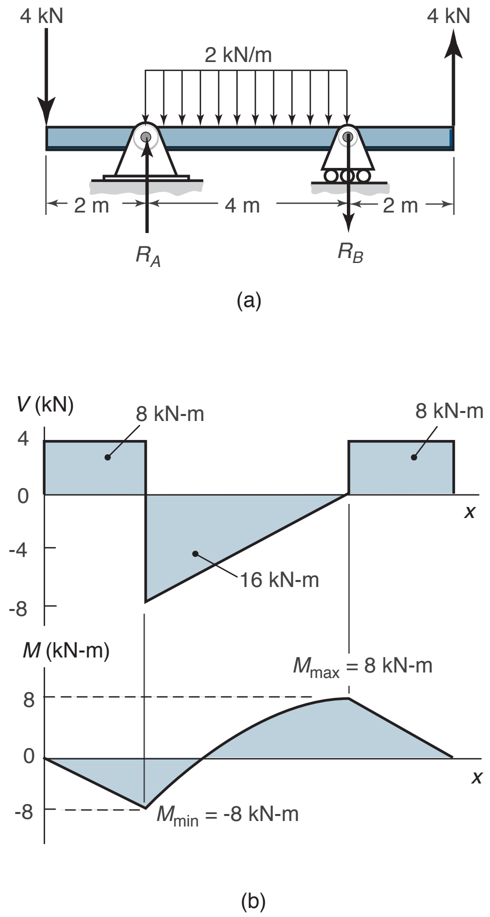


Figure 2.10: Beam for Example 2.8. (a) Applied loads and reactions; (b) shear diagram with areas indicated, and moment diagram with maximum and minimum values indicated.

3. The constant distributed loading to the right of $x = 2$ m will result in a shear force that changes linearly with respect to x. From Eq. (2.4), the magnitude of the total change is the integral of the applied load, or just its area. Thus, the total change due to the 2kN/m distributed load from $x = 2$ to $x = 6$ is 8 kN, and since the distributed load acts downward, this change is upward in the shear diagram because of the sign convention used in Eq. (2.4). Therefore, the value of the shear force at $x = 6$ is $(-8\text{ kN}) + 8\text{ kN} = 0$. The line from $x = 2$ to $x = 6$ is shown.

4. At $x = 6$, there is a concentrated force associated with the downwards acting force R_B, so there is an upward jump of 4 kN.

5. At $x = 8$, the upward acting force leads to a downward jump of 4 kN, returning the shear to zero.

The bending moment is obtained from repeated application of Eq. (2.5). However, note that the integral of the shear force is the area under the shear force curve. The shear diagram just developed consists of rectangles and triangles, where the area is calculated from geometry. The areas have been indicated in the shear diagram. For example, the shear diagram up to $x = 2$ consists of a rectangle with a height of $V = 4$ kN and a base of $x = 2$ m. Thus, its area is 8 kN-m.

The moment diagram is then constructed using the following steps.

1. At a starting value of $M = 0$ at $x = 0$, the diagram will be constructed from left to right. From $x = 0$ to $x = 2$ m, the value of the shear diagram is positive and constant. Integrating this curve results in a linear profile. Since the shear diagram is positive, the moment that results must be negative according to Eq. (2.5), and at $x = 2$ m, the value is 8 kN-m. This linear profile is shown in the figure.

2. From $x = 2$ m to $x = 6$ m, the shear diagram is linear with respect to x, so that the moment diagram will be quadratic. At $x = 6$ m, it is known that the moment will have a value of 8 kN-m by summing the areas of the shear diagram segments. The slope of the moment curve is equal to the value of the shear curve, as seen by taking the derivative of Eq. (2.5). Thus, the slope is initially large and at $x = 6$ it is zero.

3. From $x = 6$ m to $x = 8$ m, the moment diagram has a linear profile and ends at $M = 0$. This can be seen by summing the areas in the shear diagram, remembering that areas below the abscissa are considered negative.

The shear and moment diagrams are shown in Fig. 2.10b. It can be seen that the largest magnitude of shear stress is at $x = 2$ m and has a value of $|V|_{\text{max}} = 8$ kN. The largest magnitude of bending moment is $|M|_{\text{max}} = 8$ kN-m.

2.8.3 Singularity Functions

If the loading is simple, the method for obtaining shear and moment diagrams described in Sections 2.8.1 or 2.8.2 can be used. Often, however, this is not the situation. For more complex loading, methods such as **singularity functions** can be used. A singularity function in terms of a variable, x, is written as

$$f_n(x) = \langle x - a \rangle^n. \tag{2.6}$$

where n is any integer (positive or negative) including zero, and a is a reference location on a beam. Singularity functions are denoted by using angular brackets. The advantage of using a singularity function is that it permits writing an analytical expression directly for the transverse shear and moment over a range of discontinuities.

Table 2.2 shows six singularity and load intensity functions along with corresponding graphs and expressions. Note in particular the inverse ramp example. A unit step is constructed beginning at $x = a$, and the ramp beginning at $x = a$ is subtracted. To have the negative ramp discontinued at $x = a + b$, a positive ramp beginning at this point is constructed; the summation results in the desired loading.

Design Procedure 2.3: Singularity Functions

Some general rules relating to singularity functions are:

1. If $n > 0$ and the expression inside the angular brackets is positive (i.e., $x \geq a$), then $f_n(x) = (x - a)^n$. Note that the angular brackets to the right of the equal sign in Eq. (2.6) are now parentheses.

2. If $n > 0$ and the expression inside the angular brackets is negative (i.e., $x < a$), then $f_n(x) = 0$.

3. If $n < 0$, then $f_n(x) = 0$.

4. If $n = 0$, then $f_n(x) = 1$ when $x \geq a$ and $f_n(x) = 0$ when $x < a$.

5. If $n \geq 0$, the integration rule is

$$\int_{-\infty}^{x} \langle x - a \rangle^n = \frac{\langle x - a \rangle^{n+1}}{n+1}.$$

   Note that this is the same as if there were parentheses instead of angular brackets.

6. If $n < 0$, the integration rule is

$$\int_{-\infty}^{x} \langle x - a \rangle^n dx = \langle x - a \rangle^{n+1}.$$

7. When $n \geq 1$, then

$$\frac{d}{dx}\langle x - a \rangle^n = n\langle x - a \rangle^{n-1}.$$

Design Procedure 2.4: Shear and Moment Diagrams by Singularity Functions

The procedure for drawing the shear and moment diagrams by making use of singularity functions is as follows:

1. Draw a free-body diagram with all the applied distributed and concentrated loads acting on the beam, and determine all support reactions. Resolve the forces into components acting perpendicular and parallel to the beam's axis.

2. Write an expression for the load intensity function $q(x)$ that describes all the singularities acting on the beam. Use Table 2.2 as a reference, and make sure to "turn off" singularity functions for distributed loads and the like that do not extend across the full length of the beam.

3. Integrate the negative load intensity function over the beam length to get the shear force. Integrate the negative shear force distribution over the beam length to get the moment, in accordance with Eqs. (2.4) and (2.5).

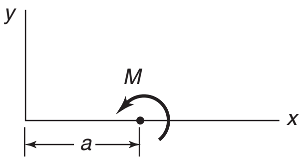
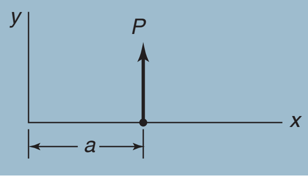
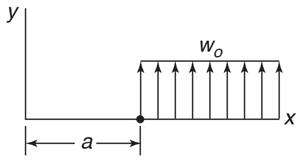
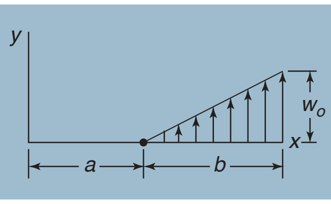
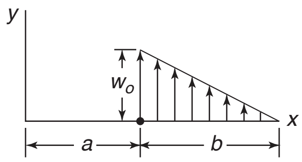
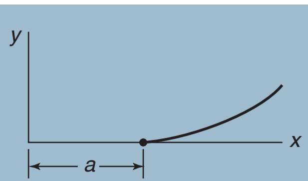
4. Draw shear and moment diagrams from the expressions developed.

Example 2.9: Shear and Moment Diagrams Using Singularity Functions

Given: The same conditions as in Example 2.7.

Find: Draw the shear and moment diagrams by using a singularity function for a concentrated force located midway on the beam.

Table 2.2: Singularity and load intensity functions with corresponding graphs and expressions.

Singularity	**Graph of $q(x)$**	**Expression for $q(x)$**
Concentrated moment		$q(x) = M\langle x-a\rangle^{-2}$
Concentrated force		$q(x) = P\langle x-a\rangle^{-1}$
Unit step		$q(x) = w_o\langle x-a\rangle^{0}$
Ramp		$q(x) = \frac{w_o}{b}\langle x-a\rangle^{1}$
Inverse ramp		$q(x) = w_o\langle x-a\rangle^{0} - \frac{w_o}{b}\langle x-a\rangle^{1}$
Parabolic shape		$q(x) = \langle x-a\rangle^{2}$

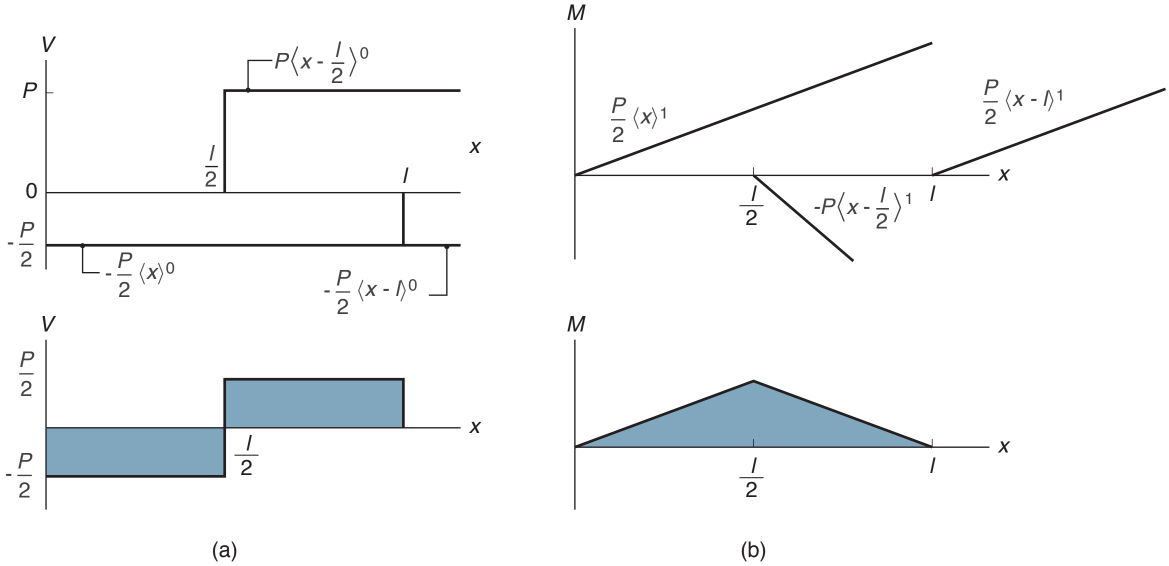


Figure 2.11: (a) Shear and (b) moment diagrams for Example 2.9.

Solution: The load intensity function for the simply supported beam shown in Fig. 2.9a is

$$q(x) = \frac{P}{2}\langle x\rangle^{-1} - P\left\langle x - \frac{l}{2}\right\rangle^{-1} + \frac{P}{2}\langle x - l\rangle^{-1}$$

The shear expression is

$$V(x) = -\int_{-\infty}^{x}\left[\frac{P}{2}\langle x\rangle^{-1} - P\left\langle x - \frac{l}{2}\right\rangle^{-1} + \frac{P}{2}\langle x - l\rangle^{-1}\right]dx$$

or

$$V(x) = -\frac{P}{2}\langle x\rangle^{0} + P\left\langle x - \frac{l}{2}\right\rangle^{0} - \frac{P}{2}\langle x - l\rangle^{0}$$

Figure 2.11a shows the resulting shear diagrams. The diagram at the top shows individual shear, and the diagram below shows the composite of these shear components. The moment expression is

$$M(x) = -\int_{-\infty}^{x}\left[-\frac{P}{2}\langle x\rangle^{0} + P\left\langle x - \frac{l}{2}\right\rangle^{0} - \frac{P}{2}\langle x - l\rangle^{0}\right]dx$$

or

$$M(x) = \frac{P}{2}\langle x\rangle^{1} - P\left\langle x - \frac{l}{2}\right\rangle^{1} + \frac{P}{2}\langle x - l\rangle^{1}$$

Figure 2.11b shows the moment diagrams. The diagram at the top shows individual moments; the diagram at the bottom is the composite moment diagram. The slope of M_2 is twice that of M_1 and M_3, which are equal. The resulting shear and moment diagrams are the same as those found in Example 2.7.

Example 2.10: Shear and Moment Expressions Using Singularity Functions

Given: A simply supported beam shown in Fig. 2.12a where $P_1 = 8$ kN, $P_2 = 5$ kN, $w_o = 4$ kN/m, and $l = 12$ m.

Find: The shear and moment expressions as well as their corresponding diagrams while using singularity functions.

Solution: The first task is to solve for the reactions at $x = 0$ and $x = l$. The force representation is shown in Fig. 2.12b. Note that w_o is defined as the load per unit length for the central part of the beam. In Fig. 2.12b it can be seen that the unit step w_o over a length of $l/2$ produces a resultant force of $w_o l/2$ and that the positive ramp over the length of $l/4$ can be represented by a resultant vector of

$$w_o\left(\frac{l}{4}\right)\left(\frac{1}{2}\right) \quad \text{or} \quad \frac{w_o l}{8}$$

Also, note that the resultant vector acts at

$$x = \left(\frac{2}{3}\right)\left(\frac{l}{4}\right) = \frac{l}{6}$$

From force equilibrium

$$0 = R_1 + P_1 + P_2 + R_2 - \frac{w_o l}{2} - \frac{w_o l}{8} \qquad (a)$$

$$R_1 + R_2 = -P_1 - P_2 + \frac{5w_o l}{8} \qquad (b)$$

Making use of moment equilibrium and the moment of the triangular section load gives

$$\frac{(P_1 + 2P_2)l}{4} - \frac{w_o l^2}{4} - \frac{w_o l}{8}\left(\frac{l}{6}\right) + R_2 l = 0$$

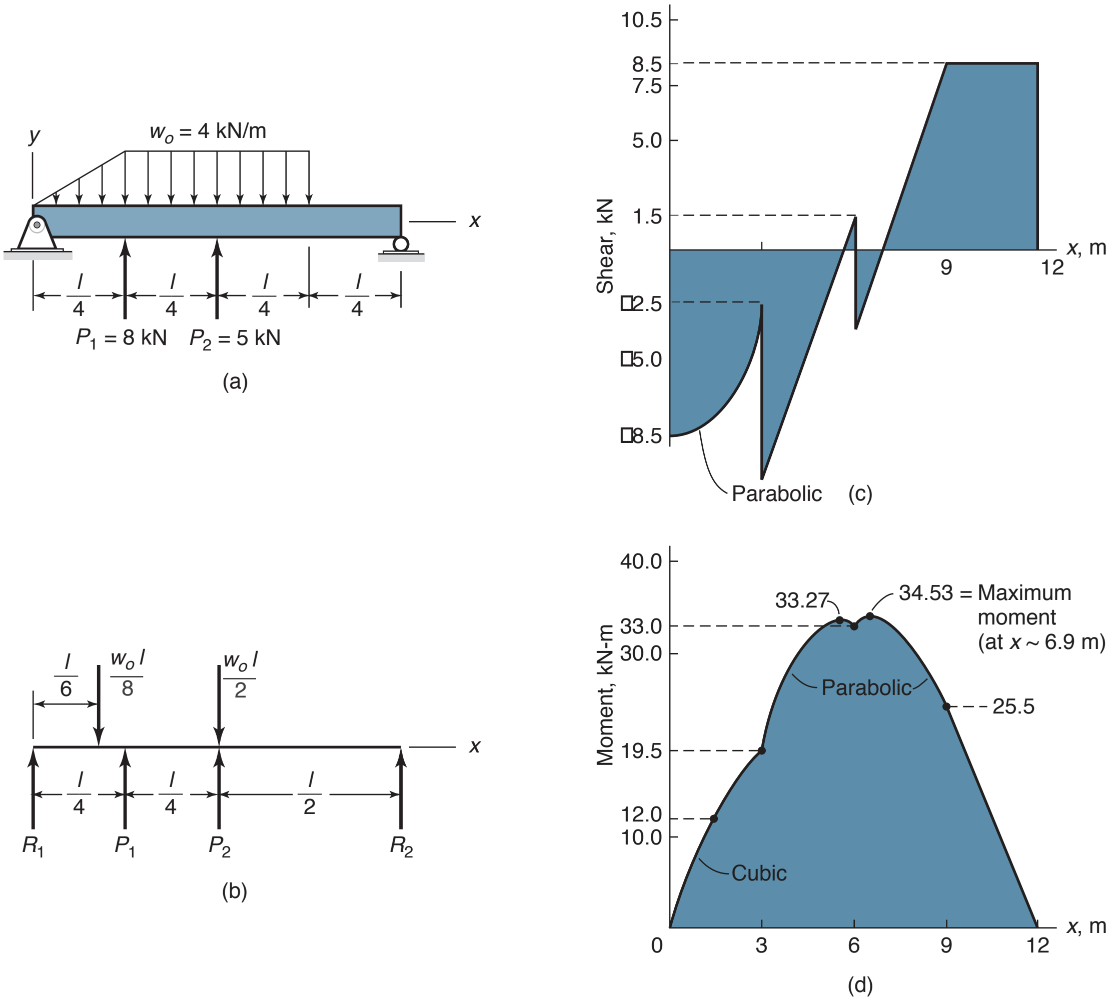


Figure 2.12: Simply supported beam examined in Example 2.10. (a) Forces acting on beam when P_1 = 8 kN, P_2 = 5 kN; w_o = 4 kN/m; l = 12 m; (b) free-body diagram showing resulting forces; (c) shear and (d) moment diagrams.

or

$$R_2 = \frac{13w_o l}{48} - \frac{P_1 + 2P_2}{4} \qquad (c)$$

Substituting Eq. (*c*) into Eq. (*b*) gives

$$R_1 = -\frac{3P_1}{4} - \frac{P_2}{2} + \frac{17w_o l}{48} \qquad (d)$$

Substituting the given values for P_1, P_2, w_o, and l gives

$$R_1 = 8.5 \text{ kN} \quad \text{and} \quad R_2 = 8.5 \text{ kN} \qquad (e)$$

The load intensity function can be written as

$$\begin{aligned} q(x) &= R_1\langle x\rangle^{-1} - \frac{w_o}{l/4}\langle x\rangle^1 + \frac{w_o}{l/4}\left\langle x - \frac{l}{4}\right\rangle^1 \\ &\quad + P_1\left\langle x - \frac{l}{4}\right\rangle^{-1} + P_2\left\langle x - \frac{l}{2}\right\rangle^{-1} \\ &\quad + w_o\left\langle x - \frac{3l}{4}\right\rangle^0 + R_2\langle x - l\rangle^{-1} \end{aligned}$$

Note that a unit step beginning at $l/4$ is created by initiating a ramp at $x = 0$ acting in the negative direction and summing it with another ramp starting at $x = l/4$ acting in the positive direction, since the slopes of the ramps are the same. The second and third terms on the right side of the load intensity function produce this effect. The sixth term on the right side of the equation turns off the unit step. Integrating the load intensity function gives the shear force as

$$\begin{aligned} V(x) &= -R_1\langle x\rangle^0 + \frac{2w_o}{l}\langle x\rangle^2 - \frac{2w_o}{l}\left\langle x - \frac{l}{4}\right\rangle^2 \\ &\quad - P_1\left\langle x - \frac{l}{4}\right\rangle^0 - P_2\left\langle x - \frac{l}{2}\right\rangle^0 \\ &\quad - w_o\left\langle x - \frac{3l}{4}\right\rangle^1 - R_2\langle x - l\rangle^0 \end{aligned}$$

Integrating the shear force gives the moment, and substituting the values for w_o and l gives

$$\begin{aligned} M(x) &= 8.5\langle x\rangle^1 + \frac{2}{9}\langle x\rangle^3 - \frac{2}{9}\langle x - 3\rangle^3 + 8\langle x - 3\rangle^1 \\ &\quad + 5\langle x - 6\rangle^1 + 2\langle x - 9\rangle^2 + 8.5\langle x - 12\rangle^1 \end{aligned}$$

The shear and moment diagrams are shown in Fig. 2.12c and d, respectively.